The Calculus Of The Normal Distribution

Gary Schurman, MBE, CFA

October, 2010

Question: We are pulling a random number from a normal distribution with a mean of 2.5 and a variance of 4.0. What is the probability that the random number will be between 1.0 and 3.0?

Legend of Symbols

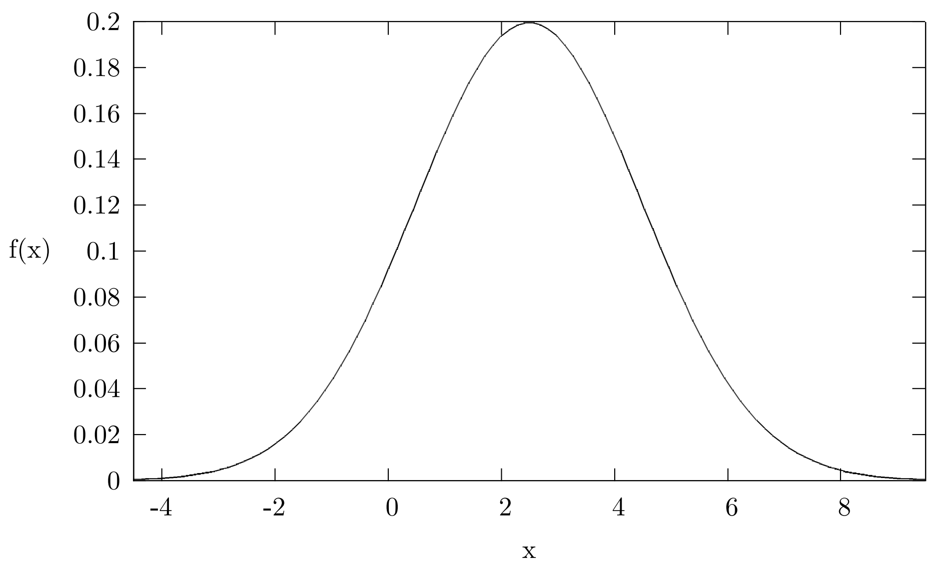
m	=	Distribution mean
v	=	Distribution variance
u	=	Upper bound of integration
l	=	Lower bound of integration
$N[z]$	=	Cumulative normal distribution function

The Calculus of the Normal Curve

The equation for the normal curve where x is the random variable (independent variable) and $f(x)$ is the height of the curve (dependent variable) is...

$$f(x) = \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(x-m)^2} \tag{1}$$

The graph below represents our normal curve where the mean is 2.5 (the variable m) and the variance is 4.0 (the variable v)...



The minimum and maximum values of the independent variable x is negative infinity and positive infinity, respectively. The area under the normal curve is the integral of equation (1) above where the lower bound of integration is negative infinity and the upper bound is positive infinity and is equal to one. The equation for the area under the normal curve is...

$$
\begin{aligned}
area &= \int_{-\infty}^{\infty} f(x)\,\delta x \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(x-m)^2}\,\delta x \\
&= 1 \qquad (2)
\end{aligned}
$$

The equation for the probability that a normally-distributed random variable x with mean m and variance v will lie in the interval $[l : u]$ is the integral of equation (1) above where the lower bound of integration is l and the upper bound of integration is u. The equation for this probability is...

$$P\Big[l \leq x \leq u\Big] = \int_{l}^{u} \frac{1}{\sqrt{2\pi v}}\, e^{-\frac{1}{2v}(x-m)^2}\,\delta x \qquad (3)$$

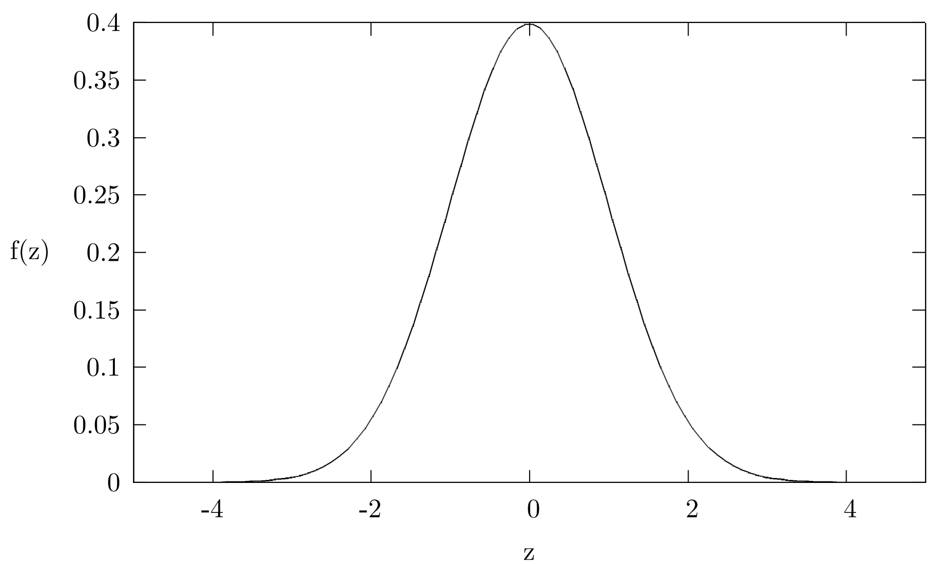
We often times want to normalize a distribution. Normalizing means that we transform a normal distribution with with mean m and variance v to a normal distribution with mean zero and variance one. We can normalize our distribution by subtracting it's mean and dividing by it's standard deviation. The normalizing equation is...

$$z = \frac{x-m}{\sqrt{v}} \qquad (4)$$

The variable z above is a new random variable that is the old random variable x minus the mean and divided by the standard deviation. The equation for our adjusted normal curve is...

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2} \qquad (5)$$

By normalizing we've moved the mean of the distribution from 2.5 to zero and we've decreased the variance from 4.0 to one. The graph below represents our normal curve where the mean is zero and the variance is one...



Note that when we normalize we must adjust the bounds of integration accordingly. The equation for the probability that a normally-distributed random variable x with mean m and variance v will lie in the interval $[l : u]$ is the integral of equation (5) above where the lower bound of integration is l and the upper bound of integration is u. The equation for this probability is...

$$P\Big[l \leq x \leq u\Big] = \int_{\frac{l-m}{\sqrt{v}}}^{\frac{u-m}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\,\delta z \qquad (6)$$

The Answer to Our Question

The cumulative standard normal distribution function, which is included in most spreadsheet packages such as Excel, measures the area under a normal curve with mean zero and variance one where the lower bound of integration is negative infinity and the upper bound is the variable of interest. We can solve our problem by calculating the probability that the random variable x will be less than 3.0 and then subtract the probability that the random variable will be less than 1.0. The equation for the cumulative standard normal distribution function is...

$$\begin{aligned} P\Bigg[x \leq \alpha\Bigg] &= \int\limits_{-\infty}^{\frac{\alpha - m}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z^2} \, \delta z \\ &= N\left[\frac{\alpha - m}{\sqrt{v}}\right] \end{aligned} \tag{7}$$

The answer to the question above is therefore...

$$\begin{aligned} P\Bigg[1 \leq x \leq 3\Bigg] &= \int\limits_{\frac{1-2.5}{\sqrt{4}}}^{\frac{3-2.5}{\sqrt{4}}} \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}z^2} \, \delta z \\ &= N\left[\frac{3 - 2.5}{\sqrt{4}}\right] - N\left[\frac{1 - 2.5}{\sqrt{4}}\right] \\ &= 0.5987 - 0.2266 \\ &= 0.3721 \end{aligned} \tag{8}$$